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ANISOTROPIC NONLINEAR STRESS–STRAIN LAWS AND YIELD CONDITIONS

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Abstract—Nonlinear stress–strain laws and yield conditions are derived for anisotropic materials on the basis of one, two and three invariants.

1. INTRODUCTION

This investigation concerns nonlinear stress–strain laws describing reversible or irreversible deformations of homogeneous anisotropic materials beyond the linear elastic range. For statically indeterminate problems such stress–strain laws give a suitable foundation for the calculation of effects caused by deviations from Hooke's law. An important effect of such kind arises for example in stress concentration problems (called by the author "macro-support effect", Neuber [28]) and leads to a considerable decrease of stress concentration even for small deviations from Hooke's law. Similar effects occur in crack propagation problems (Hutchinson [29]).

The nonlinear anisotropic stress–strain laws derived here are based on one, two and three invariants. Thermal effects may be absent. By means of the invariants introduced, yield conditions and incremental stress–strain laws for anisotropic materials are also established.

2. STRAIN ENERGY AND COMPLEMENTARY ENERGY

The strain energy density W and the complementary energy $\overline{W}$ are defined by

$$\delta W = \tau_{km}\delta_{km}, \qquad \delta\overline{W} = e_{km}\delta\tau_{km}. \tag{1}$$

Here τ_{km} denotes the stress tensor, e_{km} the strain tensor, the indices k, m and later n, p, q, r, s, t are related to the Cartesian coordinates x, y, z, and the convention of summation holds with regard to coincident indices in the same term, the sign δ denotes variations of the physical state. For materials without memory, W and $\overline{W}$ depend only on the values of e_{km} or τ_{km} (not on the history of loading) and, therefore, W and $\overline{W}$ are then potentials:

$$\tau_{km} = \partial W/\partial e_{km}, \qquad e_{km} = \partial\overline{W}/\partial\tau_{km}. \tag{2}$$

3. THE NONLINEAR ANISOTROPIC STRESS–STRAIN LAW WITH ONE INVARIANT

The simplest possibility of describing the physically nonlinear deformation of homogeneous materials consists in the introduction of one invariant factor in the stress–strain relations depending on the strength intensity. This factor may be denoted by Ψ. For the transition to Hooke's law it may have the value 1. Then the nonlinear stress–strain relations with anisotropy can be written in the following form:

$$\tau_{km} = \Psi E_{kmnp} e_{np}. \tag{3}$$

Here the tensor of rank four E_{kmnp} represents the tensor of elasticity. In consequence of the supposed homogeneity this tensor is constant and satisfies the following conditions of symmetry:

$$E_{kmnp} = E_{mknp} = E_{kmpn} = E_{npkm}. \tag{4}$$

Introducing (3) into (1) there follows

$$\delta W = \psi E_{kmnp} e_{np} \delta e_{km}, \qquad \delta \overline{W} = E_{kmnp} e_{km} \delta(\psi e_{np}). \tag{5}$$

The quasi-Hookean strain energy density W_H may be represented in the form

$$W_H = \tfrac{1}{2} E_{kmnp} e_{km} e_{np}. \tag{6}$$

Now (5) leads to

$$\delta W = \Psi \delta W_H, \qquad \delta \overline{W} = \Psi \delta W_H + 2 W_H \delta \Psi, \tag{7}$$

$$\delta W + \delta \overline{W} = 2\delta(\psi W_H), \qquad W + \overline{W} = \tau_{km} e_{km} = 2\psi W_H. \tag{8}$$

For materials with existing strain energy potential the deformation is *reversible* and the factor ψ is a function of W_H and satisfies the relation

$$\psi = \frac{\mathrm{d}W}{\mathrm{d}W_H}. \tag{9}$$

If ψ is a given function of W, then W_H can be derived as a function of W by means of the relation

$$W_H = \int_{\xi=0}^{W} \frac{\mathrm{d}\xi}{\psi(\xi)}. \tag{10}$$

The complementary energy $\overline{W} = \tau_{km} e_{km} - W$ then can be derived without contradictions as a function of W or W_H. The result shows that with an existing strain energy potential the one-invariant theory is related to the strain energy density as the only possible governing invariant characterising the strength intensity.

For irreversible deformations the strain energy potential does not exist and ψ can be a function of the three independent invariants

$$\phi_1 = A_{km} e_{km}, \qquad \phi_2 = B_{kmnp} e_{km} e_{np}, \qquad \phi_3 = C_{kmnpqr} e_{km} e_{np} e_{qr}. \tag{11}$$

Herein the tensor A_{km} is symmetric in k and m, the tensor B_{kmnp} is symmetric with regard to k and m, as well as n and p, and (k, m) and (n, p). The tensor of rank six is symmetric with regard to k and m, n and p, q and r, and (k, m) and (n, p) as well as to (n, p) and (q, r). In the *isotropic case* the tensor of elasticity E_{kmnp} is a bilinear combination of Kronecker-symbols and has the form

$$E_{kmnp} = \frac{E}{2(1+\nu)} \left[\delta_{kn}\delta_{mp} + \delta_{mn}\delta_{kp} + \frac{2\nu}{1+\nu} \delta_{km}\delta_{np} \right]. \tag{12}$$

Here E is Young's modulus and $1/\nu$ Poisson's ratio. The three invariants then can be introduced in the form

$$\phi_1 = e_{kk}, \qquad \phi_2 = e_{km}e_{km}, \qquad \phi_3 = e_{km}e_{mp}e_{kp}. \tag{13}$$

If for an isotropic material ψ is to be determined by means of the uniaxial tension test with the stress σ and the strain ε (in direction of σ), there follow $\sigma = \psi E\varepsilon$ and $W_H = E\varepsilon^2/2$. Using a diagram with σ and $E\varepsilon$ as coordinates arctan ψ represents the angle between the $E\varepsilon$-axis and the straight line leading from the origin to the point of the stress–strain line. The quantity $\psi E = E_s$ therefore can be called the *secant-modulus*. Because of its applicability to the uniaxial tension test the one-invariant theory is suitable for approximative technical strength calculations. The author used this theory to solve some stability problems of nonlinear elastic continua (Neuber [21, 22, 24, 25]).

4. THE NONLINEAR ANISOTROPIC STRESS–STRAIN LAW WITH TWO INVARIANTS

To have a more accurate representation of the nonlinear behaviour of materials two governing invariants can be introduced. It is well known that with Hooke's law for isotropy the tensors of stress, strain, elasticity and the strain energy can be represented each by two physically independent parts related to two governing invariants (the tensors of stress and strain then are to be separated into sphere tensors and deviators). Here it will be proved, that such a separation of the complete state into two physically independent states is possible for anisotropy too. Furthermore, if the possibility of such separation can be assumed for the nonlinear stress–strain law with two invariants then the theory can be represented in a very elegant form which offers considerable simplifications because of the fact that each of the two physically independent states follows a one-invariant theory. Index 1 may refer to the first, index 2 to the second invariant; then the following relations are to be satisfied (ψ_1 being a function of the first, ψ_2 a function of the second invariant, $\overset{1}{E}_{kmnp}$ and $\overset{2}{E}_{kmnp}$ being the constant tensors of elasticity of the two states of stress). Then the following conditions are to be satisfied:

$$\tau_{km} = \overset{1}{\tau}_{km} + \overset{2}{\tau}_{km}, \qquad e_{km} = \overset{1}{e}_{km} + \overset{2}{e}_{km}, \tag{14}$$

$$E_{kmnp} = \overset{1}{E}_{kmnp} + \overset{2}{E}_{kmnp}, \tag{15}$$

$$\overset{1}{\tau}_{km} = \psi_1 \overset{1}{E}_{kmnp}\overset{1}{e}_{np}, \qquad \overset{2}{\tau}_{km} = \psi_2 \overset{2}{E}_{kmnp}\overset{2}{e}_{np}, \tag{16}$$

$$W = W_1 + W_2, \qquad \overline{W} = \overline{W}_1 + \overline{W}_2, \tag{17}$$

$$W + \overline{W} = W_1 + \overline{W}_1 + W_2 + \overline{W}_2 = \tau_{km}e_{km} = \overset{1}{\tau}_{km}\overset{1}{e}_{km} + \overset{2}{\tau}_{km}\overset{2}{e}_{km}. \tag{18}$$

The transition to Hooke's law may be characterized by $\psi_1 = \psi_2 = 1$. For separating the strain tensor into its two parts the additional tensors of rank four $\overset{1}{c}_{kmnp}$ and $\overset{2}{c}_{kmnp}$ may be introduced. In consequence of the supposed homogeneity these tensors, as well as $\overset{1}{E}_{kmnp}$

and $\overset{2}{E}_{kmnp}$, are constant and symmetric as E_{kmnp} [see equation (4)]. Then the following relations hold:

$$\overset{1}{e}_{np} = \overset{1}{c}_{npqr}e_{qr}, \qquad \overset{2}{e}_{np} = \overset{2}{c}_{npqr}e_{gr}. \tag{19}$$

With regard to equation (14) the condition

$$\overset{1}{c}_{npqr}+\overset{2}{c}_{npqr} = \tfrac{1}{2}(\delta_{nq}\delta_{pr}+\delta_{nr}\delta_{pq}) \tag{20}$$

must be satisfied. Now equation (16) can be written in the form

$$\overset{1}{\tau}_{km} = \psi_1\overset{1}{E}_{kmnp}\overset{1}{c}_{npqr}e_{qr}, \qquad \overset{2}{\tau}_{km} = \psi_2\overset{2}{E}_{kmnp}\overset{2}{c}_{npqr}e_{qr}, \tag{21}$$

or using equation (20)

$$\overset{1}{\tau}_{km} = \psi_1\overset{1}{E}_{kmnp}\overset{1}{c}_{npqr}(\overset{1}{c}_{qrst}+\overset{2}{c}_{qrst})e_{st}, \qquad \overset{2}{\tau}_{km} = \dots \tag{22}$$

The identity with equation (16) is guaranteed by the conditions

$$\overset{1}{E}_{kmnp}\overset{1}{c}_{npqr} = \overset{1}{E}_{kmqr}, \qquad \overset{1}{E}_{kmnp}\overset{2}{c}_{npqr} = 0. \tag{23}$$

Therefore by the algebraic operator $\overset{1}{c}_{npqr}$ the tensor $\overset{1}{E}_{kmnp}$ is transformed into itself. The second condition is identical with the first if equation (20) is applied. Considering $\overset{2}{\tau}_{km}$ the analogous conditions

$$\overset{2}{E}_{kmnp}\overset{2}{c}_{npqr} = \overset{2}{E}_{kmqr}, \qquad \overset{2}{E}_{kmqr}\overset{1}{c}_{qrst} = 0 \tag{24}$$

can be derived. Using equation (20) at the left hand side of equation (15) there follows

$$E_{kmqr}(\overset{1}{c}_{qrnp}+\overset{2}{c}_{qrnp}) = \overset{1}{E}_{kmnp}+\overset{2}{E}_{kmnp} \tag{25}$$

and therefore

$$\overset{1}{E}_{kmnp} = E_{kmqr}\overset{1}{c}_{qrnp}, \qquad \overset{2}{E}_{kmnp} = E_{kmqr}\overset{2}{c}_{qrnp}. \tag{26}$$

Now equation (23) and (24) can be written in the form

$$E_{kmnp}\overset{1}{c}_{npqr}\overset{1}{c}_{qrst} = E_{kmqr}\overset{1}{c}_{qrst}, \dots \tag{27}$$

Multiplying the second equation (23) with E_{stqr} and using equation (26) there follows

$$\overset{1}{E}_{kmnp}\overset{2}{E}_{npqr} = 0. \tag{28}$$

This equation—together with equation (15)—is sufficient to calculate the tensors $\overset{1}{E}_{kmnp}$ and $\overset{2}{E}_{kmnp}$ if E_{kmnp} is known. The tensors $\overset{1}{c}_{kmnp}$ and $\overset{2}{c}_{kmnp}$ then can be calculated from (26). These tensors are separating not only the tensors of strain and elasticity but also the stress tensor as follows from equations (14)–(16) and equations (20)–(26):

$$\tau_{km} = (\psi_1\overset{1}{E}_{kmnp}+\psi_2\overset{2}{E}_{kmnp})e_{np}, \qquad \overset{1}{\tau}_{km} = \overset{1}{c}_{kmnp}\tau_{np}, \qquad \overset{2}{\tau}_{km} = \overset{2}{c}_{kmnp}\tau_{np}. \tag{29}$$

Using equations (12), (20), (21) and (23) the invariant $W+\overline{W} = \tau_{km}e_{km}$ can be written in the form

$$\tau_{km}e_{km} = (\psi_1 \overset{1}{E}_{kmnp}+\psi_2\overset{2}{E}_{kmnp})(\overset{1}{c}_{npqr}+\overset{2}{c}_{npqr})(\overset{1}{c}_{kmst}+\overset{2}{c}_{kmst})e_{qr}e_{st}. \tag{30}$$

The identity with the right hand side of equation (18) leads to the conditions

$$\overset{1}{E}_{kmnp}\overset{1}{c}_{npqr}\overset{2}{c}_{kmst} = 0, \qquad \overset{2}{E}_{kmnp}\overset{2}{c}_{npqr}\overset{1}{c}_{kmst} = 0, \tag{31}$$

or using (26) and (27)

$$E_{kmnp}\overset{1}{c}_{npqr}\overset{2}{c}_{kmst} = 0. \tag{32}$$

This result becomes identical with the right hand equations (23) and (24) in regard to the symmetry conditions of $\overset{1}{E}_{kmnp}$ and $\overset{2}{E}_{kmnp}$.

In the special case of *isotropy* the tensors of rank four are combinations of Kronecker symbols. With $G = E/(2+2\nu)$ as shear modulus and $1/\nu$ as Poisson's ratio they can be represented in the form:

$$\begin{aligned}
\frac{1}{G}\overset{1}{E}_{kmnp} &= \delta_{kn}\delta_{mp}+\delta_{kp}\delta_{mn}-\tfrac{2}{3}\delta_{km}\delta_{np}\\
\frac{1}{G}\overset{2}{E}_{kmnp} &= \frac{2(1+\nu)}{3(1-2\nu)}\delta_{km}\delta_{np},\\
\overset{1}{c}_{kmnp} &= \tfrac{1}{2}(\delta_{kn}\delta_{mp}+\delta_{kp}\delta_{mn})-\tfrac{1}{3}\delta_{km}\delta_{np},\\
\overset{2}{c}_{kmnp} &= \tfrac{1}{3}\delta_{kn}\delta_{mp}.
\end{aligned} \tag{33}$$

The tensors of rank four are represented in the Tables 1–5 for isotropy. As can be seen the conditions (23)–(31) are satisfied.

For anisotropy the Tables 6–10 represent a numerical example (λ being a constant factor). As can be checked easily the conditions (23)–(31) are satisfied.

For *reversible deformations* the strain energy potential exists and the energy density parts W_1 and W_2, the parts $\overline{W}_1$ and $\overline{W}_2$ of the complementary energy density and the

TABLE 1. THE TENSORS OF RANK FOUR FOR ISOTROPY: $\frac{1}{2G}E_{kmnp}$

np / km	11	12	22	23	33	31
11	$\frac{1-\nu}{1-2\nu}$	0	$\frac{\nu}{1-2\nu}$	0	$\frac{\nu}{1-2\nu}$	0
12	0	$\frac{1}{2}$	0	0	0	0
22	$\frac{\nu}{1-2\nu}$	0	$\frac{1-\nu}{1-2\nu}$	0	$\frac{\nu}{1-2\nu}$	0
23	0	0	0	$\frac{1}{2}$	0	0
33	$\frac{\nu}{1-2\nu}$	0	$\frac{\nu}{1-2\nu}$	0	$\frac{1-\nu}{1-2\nu}$	0
31	0	0	0	0	0	$\frac{1}{2}$

TABLE 2. THE TENSORS OF RANK FOUR FOR ISOTROPY: $\frac{1}{2G}\overset{1}{E}_{kmnp}$

np / km	11	12	22	23	33	31
11	$\frac{2}{3}$	0	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0
12	0	$\frac{1}{2}$	0	0	0	0
22	$-\frac{1}{3}$	0	$\frac{2}{3}$	0	$-\frac{1}{3}$	0
23	0	0	0	$\frac{1}{2}$	0	0
33	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0	$\frac{2}{3}$	0
31	0	0	0	0	0	$\frac{1}{2}$

TABLE 3. THE TENSORS OF RANK FOUR FOR ISOTROPY: $\frac{1}{2G}\overset{2}{E}_{kmnp}$

np / km	11	12	22	23	33	31
11	$\frac{1+\nu}{3(1-2\nu)}$	0	$\frac{1+\nu}{3(1-2\nu)}$	0	$\frac{1+\nu}{3(1-2\nu)}$	0
12	0	0	0	0	0	0
22	$\frac{1+\nu}{3(1-2\nu)}$	0	$\frac{1+\nu}{3(1-2\nu)}$	0	$\frac{1+\nu}{3(1-2\nu)}$	0
23	0	0	0	0	0	0
33	$\frac{1+\nu}{3(1-2\nu)}$	0	$\frac{1+\nu}{3(1-2\nu)}$	0	$\frac{1+\nu}{3(1-2\nu)}$	0
31	0	0	0	0	0	0

TABLE 4. THE TENSORS OF RANK FOUR FOR ISOTROPY: $\overset{1}{c}_{kmnp}$

np / km	11	12	22	23	33	31
11	$\frac{2}{3}$	0	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0
12	0	$\frac{1}{2}$	0	0	0	0
22	$-\frac{1}{3}$	0	$\frac{2}{3}$	0	$-\frac{1}{3}$	0
23	0	0	0	$\frac{1}{2}$	0	0
33	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0	$\frac{2}{3}$	0
31	0	0	0	0	0	$\frac{1}{2}$

TABLE 5. THE TENSORS OF RANK FOUR FOR ISOTROPY: $\overset{2}{c}_{kmnp}$

np / *km*	11	12	22	23	33	31
11	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0
12	0	0	0	0	0	0
22	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0
23	0	0	0	0	0	0
33	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0
31	0	0	0	0	0	0

TABLE 6. THE TENSORS OF RANK FOUR FOR AN EXAMPLE OF ANISOTROPY: λE_{kmnp}

np / *km*	11	12	22	23	33	31
11	16	0	-4	0	0	0
12	0	11	0	5	0	2
22	-4	0	4	0	6	0
23	0	5	0	3	0	-2
33	0	0	6	0	12	0
31	0	2	0	-2	0	12

TABLE 7. THE TENSORS OF RANK FOUR FOR AN EXAMPLE OF ANISOTROPY: $\lambda \overset{1}{E}_{kmnp}$

np / *km*	11	12	22	23	33	31
11	4	0	2	0	6	0
12	0	9	0	3	0	6
22	2	0	1	0	3	0
23	0	3	0	1	0	2
33	6	0	3	0	9	0
31	0	6	0	2	0	4

TABLE 8. THE TENSORS OF RANK FOUR FOR AN EXAMPLE OF ANISOTROPY: $\lambda \overset{2}{E}_{kmnp}$

km \ np	11	12	22	23	33	31
11	12	0	−6	0	−6	0
12	0	2	0	2	0	−4
22	−6	0	3	0	3	0
23	0	2	0	2	0	−4
33	−6	0	3	0	3	0
31	0	−4	0	−4	0	8

TABLE 9. THE TENSORS OF RANK FOUR FOR AN EXAMPLE OF ANISOTROPY: $\overset{1}{c}_{kmnp}$

km \ np	11	12	22	23	33	31
11	$\frac{5}{16}$	0	$\frac{1}{4}$	0	$\frac{3}{8}$	0
12	0	$\frac{1}{2}$	0	$-\frac{1}{4}$	0	$\frac{1}{8}$
22	$\frac{1}{4}$	0	$\frac{1}{2}$	0	0	0
23	0	$-\frac{1}{4}$	0	$\frac{3}{4}$	0	$\frac{1}{4}$
33	$\frac{3}{8}$	0	0	0	$\frac{3}{4}$	0
31	0	$\frac{1}{8}$	0	$\frac{1}{4}$	0	$\frac{3}{16}$

TABLE 10. THE TENSORS OF RANK FOUR FOR AN EXAMPLE OF ANISOTROPY: $\overset{2}{c}_{kmnp}$

km \ np	11	12	22	23	33	31
11	$\frac{11}{16}$	0	$-\frac{1}{4}$	0	$-\frac{3}{8}$	0
12	0	0	0	$\frac{1}{4}$	0	$-\frac{1}{8}$
22	$-\frac{1}{4}$	0	$\frac{1}{2}$	0	0	0
23	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	$-\frac{1}{4}$
33	$-\frac{3}{8}$	0	0	0	$\frac{1}{4}$	0
31	0	$-\frac{1}{8}$	0	$-\frac{1}{4}$	0	$\frac{5}{16}$

corresponding parts W_{H1} and W_{H2} of the quasi-Hookean strain energy density W_H satisfy the following relations which are quite analogous to (5)–(10):

$$W_{H1} = \tfrac{1}{2}\overset{1}{E}_{kmnp}\overset{1}{e}_{km}\overset{1}{e}_{np}, \qquad W_{H2} = \tfrac{1}{2}\overset{2}{E}_{kmnp}\overset{2}{e}_{km}\overset{2}{e}_{np},$$

$$\begin{aligned} \overset{1}{\tau}_{km}\delta\overset{1}{e}_{km} &= \psi_1\delta W_{H1}, \qquad \overset{2}{\tau}_{km}\delta\overset{2}{e}_{km} = \psi_2\delta W_{H2}, \\ \overset{1}{e}_{km}\delta\overset{1}{\tau}_{km} &= \psi_1\delta W_{H1} + 2W_{H1}\delta\psi_1, \qquad \overset{2}{e}_{km}\delta\overset{2}{\tau}_{km} = \ldots, \end{aligned} \tag{34}$$

$$\psi_1 = \mathrm{d}W_1/\mathrm{d}W_{H1}, \qquad \psi_2 = \mathrm{d}W_2/\mathrm{d}W_{H2},$$

or, if ψ_1 is a known function of W_1 and ψ_2 a known function of W_2,

$$W_{H1} = \int_{\mu=0}^{W_1} \frac{\mathrm{d}\mu}{\psi_1(\mu)}, \qquad W_{H2} = \int_{\varrho=0}^{W_2} \frac{\mathrm{d}\varrho}{\psi_2(\varrho)}. \tag{35}$$

As is to be seen from equation (32), for isotropy the tensors of stress and strain are to be separated in deviators and spherical tensors. This result was already derived by Kauderer [1, 11] and more generally by Wegner [16].

Since the state of nonlinear antiplane shear for isotropy [2, 6, 12, 13, 17, 18, 26, 27, 31] only contains deviatoric components of the stress and strain tensors, it can be represented by means of the one- or two-invariants relations (in the second case all terms with index 2 vanish). The advantages of the here derived stress–strain relations with two invariants are to be seen in two facts:

(1) The nonlinear behaviour can be represented by the two one-parametric functions $\psi_1 = \psi_1(W_{H1})$ and $\psi_2 = \psi_2(W_{H2})$.

(2) The tensors of anisotropic elasticity are determined by the linear elastic behaviour.

If the possibility of energy separation is not assumed, a more general two-invariants stress–strain law can be derived from the results of the next chapter by putting $\phi_3 = \bar{\phi}_3 = D_3 = \bar{D}_3 = 0$.

For *irreversible deformation* the relations derived here can also be used, but then ψ_1 and ψ_2 can be any functions of the three invariants given in equations (11) and (13).

5. THE NONLINEAR ANISOTROPIC STRESS–STRAIN LAW WITH THREE INVARIANTS

For *reversible anisotropic deformation* in the general case the strain energy potential density exists and depends on the three invariants:

$$W = W(\phi_1, \phi_2, \phi_3). \tag{36}$$

Then from (2) it follows

$$\tau_{km} = \sum_{\lambda=1,2,3} D_\lambda\, \partial\phi_\lambda/\partial e_{km} \quad \text{with} \quad D_\lambda = \partial W/\partial\phi_\lambda \tag{37}$$

and therefore

$$\partial D_\lambda/\partial\phi_\mu = \partial D_\mu/\partial\phi_\lambda, \qquad \lambda, \mu \equiv 1, 2, 3. \tag{38}$$

The three invariants may be introduced again in the suitable form (11) or (13).

From equations (36) and (11) the stress-strain relations for *anisotropy* are obtained in the following form:

$$\tau_{km} = D_1 A_{km} + 2D_2 B_{kmnp} e_{np} + 3D_3 C_{kmnpqr} e_{np} e_{qr}. \tag{39}$$

For *reversible isotropic deformation* the corresponding equations are obtained with regard to equation (13):

$$\tau_{km} = D_1 \delta_{km} + 2D_2 e_{km} + 3D_3 c_{kq} e_{mq}. \tag{40}$$

For describing the behaviour of materials according to special experimental results sometimes the complementary energy density instead of the strain energy density may be useful. Introducing the complementary energy density the second equation (2) must be applied and—consequently—in all equations of this chapter W, ϕ_λ, D_λ, A_{km}, B_{kmnp}, C_{kmnpqr}, τ_{km} and e_{km} are to be replaced by $\overline{W}$, $\bar{\phi}_\lambda$, $\overline{D}_\lambda$, $\overline{A}_{km}$, $\overline{B}_{kmnp}$, $\overline{C}_{kmnpqr}$, e_{km} and τ_{km}, respectively. The single paths are in complete analogy to the foregoing procedure; therefore the result may be represented without further comments:

For *reversible anisotropic deformation*:

$$\overline{W} = \overline{W}(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3),$$

$$e_{km} = \sum_\lambda \overline{D}_\lambda\, \partial\bar{\phi}_\lambda/\partial\tau_{km} \quad \text{with} \quad \overline{D}_\lambda = \partial\overline{W}/\partial\bar{\phi}_\lambda,$$

and therefore $\partial\overline{D}_\lambda/\partial\bar{\phi}_\mu = \partial\overline{D}_\mu/\partial\bar{\phi}_\lambda$,

$$\bar{\phi}_1 = \overline{A}_{km}\tau_{km}, \qquad \bar{\phi}_2 = \overline{B}_{kmnp}\tau_{km}\tau_{np}, \qquad \bar{\phi}_3 = \overline{C}_{kmnpqr}\tau_{np}\tau_{qr},$$

$$e_{km} = \overline{D}_1\overline{A}_{km} + 2\overline{D}_2\overline{B}_{kmnp}\tau_{np} + 3\overline{D}_3\overline{C}_{kmnpqr}\tau_{np}\tau_{qr}. \tag{41}$$

For *reversible isotropic deformation*:

$$\bar{\phi}_1 = \tau_{kk}, \qquad \bar{\phi}_2 = \tau_{km}\tau_{km}, \qquad \bar{\phi}_3 = \tau_{km}\tau_{mq}\tau_{kq}$$

$$e_{km} = \overline{D}_1\delta_{km} + 2\overline{D}_2\tau_{km} + 3\overline{D}_3\tau_{kq}\tau_{mq}, \qquad \partial\overline{D}_\lambda/\partial\bar{\phi}_\mu = \partial\overline{D}_\mu/\partial\bar{\phi}_\lambda. \tag{42}$$

For *irreversible deformation* the factors D_λ and $\overline{D}_\lambda$, respectively, are independent functions of the three invariants. If D_1 is proportional to ϕ_1, D_2 is constant and D_3 equal to zero, then Hooke's law is realised (also if $\overline{D}_1$ is proportional to $\bar{\phi}_1$, $\overline{D}_2$ is constant and $\overline{D}_3$ zero). If $\overline{D}_1$ is a linear combination of the three invariants and $\overline{D}_2$ and $\overline{D}_3$ are constants (irreversible deformation) the isotropic relations correspond to those used by Evans and Pister [23] and Orthwein [30].

6. THE YIELD CONDITION AND THE INCREMENTAL STRESS–STRAIN LAW

Using the invariants $\bar{\phi}_1$, $\bar{\phi}_2$, $\bar{\phi}_3$ the yield condition for anisotropy can be established in the following form:

$$\chi(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) = \text{const.} \tag{43}$$

with $\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3$ according to equation (41). If the deformation is assumed to be *elastic–plastic* in the usual sense the increment of the strain tensor can be written in the following form:

$$\delta e_{km} = H_{kmpq}\delta\tau_{pq} + L_{km}\delta A \tag{44}$$

with

$$H_{kmpq} = \sum_{\lambda=1,2,3} \frac{\partial}{\partial \tau_{pq}}\left(\frac{\partial \overline{W}}{\partial \overline{\Phi}_\lambda}\frac{\partial \overline{\Phi}_\lambda}{\partial \tau_{km}}\right) = \sum_{\lambda=1,2,3}\left(\frac{\partial \overline{W}}{\partial \overline{\Phi}_\lambda}\frac{\partial^2 \overline{\Phi}_\lambda}{\partial \tau_{km}\partial \tau_{pq}} + \sum_{\mu=1,2,3}\frac{\partial^2 \overline{W}}{\partial \overline{\Phi}_\lambda \partial \overline{\Phi}_\mu}\frac{\partial \overline{\Phi}_\lambda}{\partial \tau_{km}}\frac{\partial \overline{\Phi}_\mu}{\partial \tau_{pq}}\right) \tag{45}$$

and

$$L_{km} = \sum_{\lambda=1,2,3}\frac{\partial \chi}{\partial \overline{\Phi}_\lambda}\frac{\partial \overline{\Phi}_\lambda}{\partial \tau_{km}}. \tag{46}$$

This representation includes the known yield conditions and incremental stress–strain laws (see Refs. 3–5, 8–11, 14, 16, 19, 20, 23, 30 and 32).

If the influence of the third invariant can be neglected the terms with $\overline{\Phi}_3$ are to be eliminated. Following the procedure of chapter 4 the quasi-Hookean energy densities, written as bilinear functions of the stress components, can be introduced instead of $\overline{\Phi}_1$ and $\overline{\Phi}_2$ as governing invariants: $\overline{\Phi}_1 = W_{H1}$, $\overline{\Phi}_2 = \overline{W}_{H2}$. Then there follows $\overline{W} = \overline{W}_1(W_{H1}) + \overline{W}_2(W_{H2})$, $\chi = \chi_1(W_{H1}) + \chi_2(W_{H2}) = \text{const.}$

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Абстракт—Получаются нелинейные законы напряжение-деформация для анизотропных материалов на основе одного, двух и трех инвариантов.